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Applications of exterior difference systems to variations in discrete mechanics*

Zheng Xie and Hongbo Li

Key Laboratory of Mathematics Mechanization, Chinese Academy of Sciences, Beijing 100080, People's Republic of China

E-mail: lenozhengxie@yahoo.com.cn

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Abstract
In discrete mechanics, difference equations describe the fundamental physical laws and exhibit many geometric properties. Can these equations be obtained in a geometric way? Using some techniques in exterior difference systems, we investigate the discrete variational problem. As an application, we give a positive answer to the above question for the discrete Newton's, Euler–Lagrange, and Hamilton's equations.

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1. Introduction

In recent years, there has been a substantial growth of interest in discrete mechanics [2–12]. In this renascent field, difference equations describe the fundamental physical laws and exhibit many geometric properties such as the desirable symmetry and conservation laws. It should be an interesting problem to deduce these equations in a geometric way. In the continuous case, it is well known that utilizing techniques from exterior differential systems such as the derived flag and prolongation allows a systematic treatment of the variational principles in greater generality than customary and sheds new light on even the classical Lagrange problem [1]. Naturally, we consider how to apply the techniques in discrete differential geometry and exterior difference systems [12–17] to the discrete variations in discrete mechanics.

- Using some techniques in exterior difference systems, we set up the problem of the discrete variation on a regular lattice, deduce the discrete variational equations and obtain the discrete Euler–Lagrange and Newton's equations in a geometric way. (Section 3).
- By discrete variational equations, we obtain the discrete Hamilton's equations and Noether's theorem, which is equivalent to discrete Euler–Lagrange equations under the discrete Legendre transform. (Section 4).

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- If the variable of variation is continuous, the equations and transform here fit in the framework developed by HY Guo, K Wu *et al* [12]. The discrete Euler–Lagrange equations and Noether's theorem here also fit in the framework developed by Marsden and West *et al* [10].

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2. Preliminaries

In this section, we recall some concepts in exterior difference systems [10, 12–17], which are used in this paper.

2.1. Exterior difference operator

Consider a regular lattice Z^m with coordinates $\{x^1, \ldots, x^m\}$, where Z is a ring of integers. The *discrete tangent space* at the node $p \in Z^m$ is

$$T_p Z^m := \text{span}\{\Delta_i|_p, i = 1, \ldots, m\},$$

where Δ_i is a difference operator in the direct of x^i, such that

$$\Delta_i g(x^1, \ldots, x^m) = E_i g(x^1, \ldots, x^m) - g(x^1, \ldots, x^m),$$

where

$$E_i g(x^1, \ldots, x^m) = g(x^1, \ldots, x^i + 1, \ldots, x^m),$$

and g is a R-valued function on Z^m and R is a field of real numbers.

The *discrete cotangent space* at p is

$$T_p^* Z^m := \text{span}\{d_D x^i|_p, i = 1, \ldots, m\},$$

where $d_D x^i$ satisfies

$$\langle d_D x^i, \Delta_j \rangle_D := \Delta_j (x^i) = \delta^i_j.$$

The discrete tangent and cotangent bundles over Z^m are

$$TZ^m := \bigcup_{p \in Z^m} T_p Z^m, \qquad T^* Z^m := \bigcup_{p \in Z^m} T_p^* Z^m,$$

respectively. Sections on TZ^m and T^*Z^m are called discrete tangent vector fields and difference 1-forms, respectively.

As the differential case, we can construct the exterior difference form algebra [12]

$$\Omega^* = \oplus_{n \in Z} \Omega^n,$$

where Ω^n is a set of difference n-forms, generated by

$$h d_D x^{j_1} \wedge \cdots \wedge d_D x^{j_n}, \qquad j_1, \ldots, j_n \in 1, \ldots, m,$$

where h is the R-valued function on Z^m,

$$d_D x^i h = E_i h d_D x^i, \qquad d_D x^i \wedge d_D x^j = -d_D x^j \wedge d_D x^i.$$

The *exterior difference operator* $d_D : \Omega^k \to \Omega^{k+1}$ is defined as

$$d_D w = \sum_{i=1}^{m} \Delta_i h d_D x^i \wedge d_D x^{j_1} \wedge \cdots \wedge d_D x^{j_k},$$

where $w = h d_D x^{j_1} \wedge \cdots \wedge d_D x^{j_k}$. The d_D satisfies the Leibnitz law and $d_D^2 = 0$ [12].

2.2. Exterior difference systems

Let (α, π, Z^m) be a discrete vector bundle, i.e. $\alpha = Z^m \times R^n$ and $\pi(\alpha) = Z^m$. Suppose x^i and u^j are the coordinates on the regular lattices Z^m and R^n, respectively. Consider section f:

$$u^i = f^i(x^1, \ldots, x^m), \qquad 1 \leqslant i \leqslant n.$$

Define the map $f^* : \wedge T^* f(Z^m) \to \wedge T^* Z^m$ as follows:

$f^*(h(u^1, \ldots, u^n) d_D u^{i_1}) := h \circ f(x^1, \ldots, x^m) d_D(u^{i_1} \circ f)$,

$f^*(h(u^1, \ldots, u^n) d_D u^{i_1} \wedge \cdots \wedge d_D u^{i_r}) := h \circ f(x^1, \ldots, x^m) f^* d_D u^{i_1} \wedge \cdots \wedge f^* d_D u^{i_r}$.

f^* is linear map and commutes with $\wedge$ and d_D, called the discrete cotangent map of f [17].

Definition 2.1. *Let (α, π, Z^m) be a discrete vector bundle and any section on Z^m has the coordinate expression*

$$u^i = u^i(x^1, \ldots, x^m), \qquad 1 \leqslant i \leqslant n.$$

Let $\Omega^* = \oplus_{k \in Z} \Omega^k$, where Ω^k is a set of difference k-forms, generated by any k elements in $\{d_D x^1, \ldots, d_D x^m, d_D u^1, \ldots, d_D u^n\}$ multiply by $\wedge$, with coefficients of R-valued function on Z^m.

(1) A subring of $I \subset \Omega^*$ is called a right ideal, if
  (a) $\alpha \in I$ implies $\alpha \wedge \beta \in I$ for all $\beta \in \Omega^*$;
  (b) $\alpha \in I$ implies that all its components in Ω^* are contained in I.

(2) An exterior difference system is given by a right ideal $I \subset \Omega^*$ that is closed under d_D.

(3) An integral lattice of the system is given by a section $f : Z^m \to Z^m \times R^n$ such that $f^*\alpha = 0$ for all $\alpha \in I$.

We note that the exterior difference system used here is a local system. This system can include all the local ordinary and partial difference equations on a regular lattice, if introducing the discrete jet bundle on the regular lattice.

Definition 2.2. *Let $(\alpha, \pi, Z^m) = \{x^1, \ldots, x^m, u^1, \ldots, u^n\}$ be a discrete vector bundle and $\Delta^k_{i_1 \cdots i_k} = \Delta_{i_1} \cdots \Delta_{i_k}$. The discrete k-jet bundle of α is a discrete vector bundle with coordinates*

$$\left\{x^i, u^j, \Delta_i u^j, \ldots, \Delta^k_{i_1 \cdots i_k} u^j\right\}, \qquad 1 \leqslant i, i_1, \ldots, i_k \leqslant m, \qquad 1 \leqslant j \leqslant n,$$

denoted by $J^k_D \alpha$.

Example 2.3. Consider the second-order difference equations in the discrete vector bundle $Z \times R = \{x, y\}$,

$$\Delta^2_x y = F(x, y, \Delta_x y).$$

It can be written as

$$d_D y - \dot{y} d_D x = 0$$

$$d_D \dot{y} - F(x, y, \dot{y}) d_D x = 0$$

in $J^1_D(Z \times R) = \{x, y, \dot{y}\}$, $\dot{y} = \Delta_x y$.

Consider the partial difference equation on $Z^n \times R = \{x^1, , \ldots, x^n, z\}$,

$$F(x^i, z, \Delta_i z) = 0, \qquad 1 \leqslant i \leqslant n.$$

Letting $p_i := \Delta_i z$, it can be written as

$$F(x^i, z, p_i) = 0$$

$$d_D z - p_i d_D x^i = 0,$$

in $J^1_D(Z^n \times R) = \{x^1, \ldots, x^n, z, p_1, \ldots, p_n\}$.

2.3. Pairing formula

Consider discrete vector fields and difference 1-forms on $Z^m = \{x^1, \ldots, x^m\}$:

$$v_j = a^{k_j}\Delta_{k_j}, \qquad v^{*i} = f_{k_i} d_D x^{k_i}, \qquad 1 \leqslant i, j, k_i, k_j \leqslant m.$$

The pairing formula of $\wedge T Z^m$ and $\wedge T^* Z^m$ is

$$\langle v^{*1} \wedge \cdots \wedge v^{*p}, v_1 \wedge \cdots \wedge v_p \rangle_D := f_{i_1} E_{i_1} f_{i_2} \cdots E_{i_1 + \cdots + i_{p-1}} f_{i_p} \times \left(\sum_{\sigma} \varepsilon \delta^{i_1}_{\sigma j_1} \cdots \delta^{i_p}_{\sigma j_p} \right) a^{j_1} E_{j_1} a^{j_2} \cdots E_{j_1 + \cdots + j_{p-1}} a^{j_p},$$

where

$$\varepsilon = \begin{cases} 1, & \sigma \text{ is even arrange} \\ -1, & \sigma \text{ is odd arrange,} \end{cases} \quad E_{j_1 + \cdots + j_{p-1}} = E_{j_1} \circ \cdots \circ E_{j_{p-1}}.$$

For example, if $p = 2$, then

$$\langle f_{i_1} d_D x^{i_1} \wedge f_{i_2} d_D x^{i_2}, a^{j_1}\Delta_{j_1} \wedge a^{j_2}\Delta_{j_2} \rangle_D = f_{i_1} E_{i_1} f_{i_2} a^{i_1} E_{i_1} a^{i_2} - f_{i_1} E_{i_1} f_{i_2} a^{i_2} E_{i_2} a^{i_1}.$$

Now we can define the discrete tangent map $f_* : \wedge T Z^m \to \wedge T f(Z^m)$ as follows:

$$\left\langle h d_D u^{i_1} \wedge \cdots \wedge d_D u^{i_r}, f_*\left(k\Delta_{\alpha_1} \wedge \cdots \wedge \Delta_{\alpha_r}\right)\right\rangle_D := \langle f^*(h d_D u^{i_1} \wedge \cdots \wedge d_D u^{i_r}), k\Delta_{\alpha_1} \wedge \cdots \wedge \Delta_{\alpha_r} \rangle_D.$$

In the similar way as Beauce *et al* did [13], we can define the *discrete contract operator* i_Y:

$$\langle i_Y w, X_1 \wedge \cdots \wedge X_{r-1} \rangle_D := \langle w, X_1 \wedge \cdots \wedge X_{r-1} \wedge Y \rangle_D,$$

where $Y = Y^i \Delta_i \in T Z^m$, and *discrete Lie derivative operator* using the Cartan formula

$$L_X \omega := i_X d_D \omega + d_D i_X \omega.$$

More information about these or the similar operators can be found in [10, 12–17].

3. Discrete variational equations

In this section, we investigate the application of exterior difference systems to discrete variations. At first, we set up the problem of the discrete variation using the language of exterior difference systems.

3.1. Discrete variational problem

We consider an exterior difference system I on the discrete vector bundle $Z \times R^n$ with coordinates $\{t, q^1, \ldots, q^n\}$, where $I = \{\theta^1, \ldots, \theta^k\}$ is a set of difference 1-forms.

Giving a difference 1-form φ on $Z \times R^n$, and for each integral lattice $f(Z)$ of I, we set

$$\Phi(Z, f) = \sum_{t \in Z} \langle f^* \varphi, \Delta_t \rangle_D.$$

We may view the $\Phi : V(I) \to R$ as a function on the lattice Z, where $V(I)$ is the set of the integral lattice of I, and consider

Problem 3.1. Determine the discrete variational equations of the Φ over Z.

We denote by (I, φ) the discrete variational problem associated with the function $\Phi(Z, f)$.

Example 3.2. Let $J_D^k(Z \times R^n)$ be a discrete k-jet bundle and L be a function on $J_D^k(Z \times R^n)$. Set $\varphi = L d_D t$ and take

$I = \left\{d_D q^\alpha - \Delta_t q^\alpha d_D t, d_D \Delta_t q^\alpha - \Delta_t^2 q^\alpha d_D t, \cdots, d_D \Delta_t^{k-1} q^\alpha - \Delta_t^k q^\alpha d_D t\right\}.$

The $(I; \varphi)$ is the kth order discrete variational problem.

3.2. Discrete variational equations

Now, we follow PA Griffiths' method in the differential case [1] to derive the discrete variational equations for the integral lattice of (I, ω).

Setting a regular sublattice $[0, \infty)$ with a coordinate s, a discrete variation of f is given by

$$F : Z \times [0, \infty) \to (Z \times [0, \infty)) \times R^n, \tag{1}$$

such that if we let $f_s : Z \to Z \times R^n$ be the restriction of F to $Z \times \{s\}$, then $f_0 = f$. The associated *discrete variational vector field* is

$$V := F_*(\Delta_s)|_{\mathrm{im} f}.$$

Proposition 3.3. *Suppose that $f : Z \to Z \times R^n$ with a discrete variation F and an associated discrete variational vector field V. Let θ be a difference form on $(Z \times [0, \infty)) \times R^n$. Then*

$$L_{\Delta_s}(F^*\theta)|_Z = f^*(i_V d_D \theta + d_D i_V \theta).$$

Proof. Let $\{t, s\}$ be the coordinates of regular sublattice $Z \times [0, \infty)$ and V be a discrete vector field on $\mathrm{im} f$, such that

$$V(t) = F_* \Delta_s|_{f(t)} \in T_{\mathrm{im} f}(\mathrm{im}\, F).$$

The definition of the discrete Lie derivative operator implies that

$$\begin{aligned} L_{\Delta_s} F^*\theta &= d_D\left(i_{\Delta_s} F^*\theta\right) + i_{\Delta_s}(d_D F^*\theta) \\ &= d_D\left(i_{F_*\Delta_s}\theta\right)(t, s) + \left(i_{F_*\Delta_s} d_D \theta\right)(t, s) \\ &= F^*\left(d_D i_{F_*\Delta_s}\theta + i_{F_*\Delta_s} d_D \theta\right). \end{aligned}$$

Both sides of this equation are difference forms on $Z \times [0, \infty)$, and the proposition follows by restricting both sides to Z. □

If θ does not contain $d_D s$, then

$$\begin{aligned} L_{\Delta_s}(F^*\theta)|_Z &= f^*(i_V d_D \theta + d_D i_V \theta) \\ &= f^* i_V d_D \theta \\ &= \Delta_s(F^*\theta)|_Z. \end{aligned}$$

Now, we derive the discrete variational equations for the integral lattice of I. We assume that f_s is an integral lattice of I. The map f_s can be looked as map $F(t, s)$ in (1), so $F^*\theta^\alpha = g^\alpha(t, s) d_D s$. Thus,

$$L_{\Delta_s}(F^*\theta^\alpha)|_Z = \Delta_s g^\alpha(t, 0) d_D s|_Z = 0.$$

By Proposition 3.3, this gives

$$f^*(i_V d_D \theta^\alpha + d_D i_V \theta^\alpha) = 0. \tag{2}$$

A discrete variation of Z in $Z \times R^n$ is then given by the discrete vector field V on $\mathrm{im} f$. We extend V to a discrete vector field on $\mathrm{im} F$, still denoted by V. Then $i_V d_D \theta^\alpha + d_D(i_V \theta^\alpha)$ is a 1-form on $\mathrm{im} F$ and (2) is equivalent to

$$F^*(i_V d_D \theta^\alpha + d_D i_V \theta^\alpha)|_Z = 0. \tag{3}$$

We call (2) or (3) the *discrete variational equation* of $f : Z \to Z \times R^n$ as an integral lattice of I.

Remark 3.4.

(1) (3) depends only on V and not on the extension of V to a discrete vector field on $F(Z \times [0, +\infty))$.

(2) The (3) vanishes in case V is tangent to $f(Z)$, and therefore depends only on the section $[V]$ of the normal bundle determined by V.
In fact, suppose that V is tangent to $f(Z)$, i.e., $V = F_*\Delta_t$. If V is any extension, then $i_V\theta^\alpha|_{f(Z)} = 0$, so $F^*(d_D i_V\theta^\alpha)|_Z = 0$. Meanwhile $F^*(i_V d_D\theta^\alpha)$ contains no $d_D t$, so $F^*(i_V d_D\theta^\alpha)|_Z = 0$.

(3) Let $\theta = \theta^\alpha\lambda_\alpha$, where the λ^α are real functions on Z. Then

$$\begin{aligned} f^*(i_V d_D\theta + d_D i_V\theta) &= f^*(i_V(d_D\theta^\alpha\lambda_\alpha + \theta^\alpha d_D\lambda_\alpha) + d_D i_V(\theta^\alpha\lambda_\alpha)) \\ &= f^*(i_V d_D\theta^\alpha + d_D i_V\theta^\alpha)\lambda_\alpha \\ &= 0. \end{aligned}$$

3.3. Discrete Euler–Lagrange equations

Let V be a discrete tangent vectors field on $\operatorname{im} f$, satisfies (3) and $V|_{\{-\infty,\infty\}} = 0$. Let $Z \times \{s\}$ be a 1-parameter family of the lattice with the discrete variational vector V. By Proposition 3.3, we have

$$\begin{aligned} \Delta_s\left(\sum_{t\in Z\times\{s\}}\langle F^*\varphi, \Delta_t\rangle_D\right)\Bigg|_{s=0} &= \sum_{t\in Z}\langle f^*(i_V d_D\varphi + d_D i_V\varphi), \Delta_t\rangle_D \\ &= \sum_{t\in Z}\langle f^* i_V d_D\varphi, \Delta_t\rangle_D + i_V\varphi|_\infty - i_V\varphi|_{-\infty} \\ &= \sum_{t\in Z}\langle f^* i_V d_D\varphi, \Delta_t\rangle_D. \end{aligned}$$

We consider a function

$$\delta_D\Phi(Z, f)(V) := \Delta_s\left(\sum_{t\in Z\times\{s\}}\langle F^*\varphi, \Delta_t\rangle_D\right)\Bigg|_{s=0} = \sum_{t\in Z}\langle f^*(i_V d_D\varphi), \Delta_t\rangle_D. \tag{4}$$

Concerning (4) we make following

Remark 3.5.

(1) If we set

$$\varphi_1 = \varphi + \theta^\alpha\lambda_\alpha, \tag{5}$$

then since $\theta^\alpha|_{f(Z)} = 0$ the function $\Phi(Z, f)$ remains unchanged. So is the function $\delta_D\Phi(Z, f)$, since

$$\begin{aligned} \sum_{t\in Z}\langle f^*(i_V d_D(\varphi_1 - \varphi)), \Delta_t\rangle_D &= \sum_{t\in Z}\langle f^*(i_V d_D(\theta^\alpha\lambda_\alpha)), \Delta_t\rangle_D \\ &= -\sum_{t\in Z}\langle f^* d_D(i_V\theta^\alpha)\lambda_\alpha, \Delta_t\rangle_D \\ &= f^*(i_V\theta^\alpha)E_t\lambda_\alpha|_{-\infty} - f^*(i_V\theta^\alpha)E_t\lambda_\alpha|_\infty \\ &= 0. \end{aligned}$$

(2) If we set

$$\varphi_2(V) = \varphi + d_D\eta(V), \tag{6}$$

where $\eta(V)$ linear depends on V, then because $d_D^2 = 0$ certainly

$$\sum_{t\in Z}\langle f^*(i_V d_D\varphi), \Delta_t\rangle_D = \sum_{t\in Z}\langle f^*(i_V d_D\varphi_2), \Delta_t\rangle_D.$$

Since

$$\sum_{t\in Z}\langle f^*(\varphi_2), \Delta_t\rangle_D = \sum_{t\in Z}\langle f^*(\varphi), \Delta_t\rangle_D + f^*\eta(V)|_\infty - f^*\eta(V)|_{-\infty},$$

we shall only want to consider substitutions (6), where $\eta(V)$ depends linearly on V. For such an η we have

$$\eta_{\{-\infty,\infty\}} = 0, \tag{7}$$

whenever $V_{\{-\infty,\infty\}} = 0$ holds.

(3) The quantity of $\delta_D\Phi(Z, f)(V)$ depends only on $V \in T_{\mathrm{im} f}(\mathrm{im}F)$ and not on the extension of V. As the proof of Remark 3.4(2), if V is tangent to $f(Z)$, then $\delta_D\Phi(Z, f)(V) = 0$. Therefore $\delta_D\Phi(Z, f)(V)$ depends only on $[V]$.

Remark 3.5 follows that whenever the equations

$$\delta_D\Phi(Z, f)[V] = 0 \tag{8}$$

holds, they must be invariant under substitutions (5) and (6).

Invariance under (6) means essentially that equations (8) should be expressed in terms of $d_D\varphi$, and combining this with invariance under (5) gives the conclusion: *the (8) should be expressed in terms of $d_D(\varphi + \theta^\alpha\lambda_\alpha)$, where λ_α are to be determined real functions on Z.*

With these observations as guide, we can turn to the derivation of the discrete Euler–Lagrange equations.

Let

$$L^\alpha(V) = i_V d_D\theta^\alpha + d_D i_V\theta^\alpha. \tag{9}$$

If (8) holds, then $f^*L^\alpha(V) = 0$ and $V_{\{-\infty,\infty\}} = 0$ can induce

$$\sum_{t\in Z}\langle f^*(i_V d_D\varphi), \Delta_t\rangle_D = 0.$$

Letting $\eta(t) = \sum_{-\infty}^{t-1}\langle i_V d_D\varphi, f_*\Delta_t\rangle_D$, we have

$$i_V d_D\varphi|_{f(Z)} = d_D\eta|_{f(Z)}, \qquad \eta|_{\{-\infty,\infty\}} = 0.$$

In particular, if

$$f^*(i_V d_D\varphi) = f^*(L^\alpha(V))\lambda_\alpha + f^*d_D\eta, \tag{10}$$

then (8) will hold.

If we set $f^*\eta = -f^*(i_V\theta^\alpha)\lambda_\alpha$, then (7) is satisfied and

$$f^*(L^\alpha(V))\lambda_\alpha + f^*d_D\eta = f^*(i_V d_D(\theta^\alpha\lambda_\alpha)). \tag{11}$$

For this choice of η, and replacing λ_α by $-\lambda_\alpha$, we obtain from (10),

$$i_V(d_D(\varphi + \theta^\alpha\lambda_\alpha)) = 0|_{f(Z)}, \qquad \forall V, \tag{12}$$

or equivalent $i_V(d_D(\varphi + \theta^\alpha\lambda_\alpha))|_Z = 0$ (here we omit f^*). These equations satisfy the conditions of being invariant under substitutions (4) and (5); in fact, they are the simplest such equations. Consequently, we give the following

Definition 3.6. *The discrete Euler–Lagrange equations associated with the discrete variational problem* (I, φ) *are equations (12) on the integral lattice* $f(Z)$ *of I.*

Example 3.7. Consider discrete 1-jet bundle of $Z \times R^n = \{t, q^1, \ldots, q^n\}$ and let L be a function on $J_D^1(Z \times R^n)$.

We set $\varphi = L(t, q^\alpha, \dot{q}^\alpha) d_D t$ and take $I = \{d_D q^\alpha - \dot{q}^\alpha d_D t\}$. Then using

$$\begin{cases} \varphi = L(t, q^\alpha, \dot{q}^\alpha) d_D t \\ \theta^\alpha = d_D q^\alpha - \dot{q}^\alpha d_D t. \end{cases} \tag{13}$$

Suppose $\dot{q}^\alpha(t,s) = s + \dot{q}^\alpha(t)$ and $q^\alpha(t,s) = s + q^\alpha(t)$. Taking $V = \Delta_{\dot{q}^\alpha(t,s)}$, $V = \Delta_{q^\alpha(t,s)}$ for $i_V d_D(\varphi + \theta^\alpha \lambda_\alpha)$, respectively, the discrete Euler–Lagrange equations of (13) are

$$\begin{cases} (L_{\dot{q}^\alpha} - E_t \lambda_\alpha) d_D t = 0 \\ (L_{q^\alpha} - \Delta_t \lambda_\alpha) d_D t = 0, \end{cases} \tag{14}$$

where

$$L_{\dot{q}^\alpha} := \Delta_s L(t, q^\alpha(t), \dot{q}^\alpha(t,s))|_{s=0} = L(t, q^\alpha(t), \dot{q}^\alpha(1,t)) - L(t, q^\alpha(t), \dot{q}^\alpha(t)),$$
$$L_{q^\alpha} := \Delta_s L(t, q^\alpha(t,s), \dot{q}^\alpha(t))|_{s=0} = L(t, q^\alpha(1,t), \dot{q}^\alpha(t)) - L(t, q^\alpha(t), \dot{q}^\alpha(t)).$$

In fact when $V = \Delta_{\dot{q}^\alpha(t,s)}$, then $[V] = \Delta_s$, $d_D \dot{q}^\alpha(t,s) = d_D s +$ terms contain $d_D t$, and

$$\begin{aligned} i_V d_D(\varphi + \theta^\alpha \lambda_\alpha)|_Z &= i_{\Delta_{\dot{q}^\alpha(t,s)}} d_D(L(t, q^\alpha(t), \dot{q}^\alpha(t,s)) d_D t + (d_D q^\alpha(t) - \dot{q}^\alpha(t,s) d_D t)\lambda_\alpha)|_{s=0} \\ &= i_{\Delta_s}(L_{\dot{q}^\alpha} d_D s - d_D s E_t \lambda_\alpha) \wedge d_D t \\ &= (-L_{\dot{q}^\alpha} + E_t \lambda_\alpha) d_D t. \end{aligned}$$

If $V = \Delta_{q^\alpha(t,s)}$, then $[V] = \Delta_s$, $d_D q^\alpha(t,s) = d_D s +$ terms contain $d_D t$, and

$$\begin{aligned} i_V d_D(\varphi + \theta^\alpha \lambda_\alpha)|_Z &= i_{\Delta_{q^\alpha(t,s)}} d_D(L(t, q^\alpha(t,s), \dot{q}^\alpha(t)) d_D t + (d_D q^\alpha(t,s) - \dot{q}^\alpha(t) d_D t)\lambda_\alpha)|_{s=0} \\ &= i_{\Delta_s}(L_{q^\alpha} d_D s - d_D s \Delta_t \lambda_\alpha) \wedge d_D t \\ &= (-L_{q^\alpha} + \Delta_t \lambda_\alpha) d_D t \end{aligned}$$

Since $d_D t$ is a base of T^*Z, equations of (14) is equivalent to

$$\Delta_t E_{-t} L_{\dot{q}^\alpha} = L_{q^\alpha}. \tag{15}$$

If s is continuous variable, equation (15) is just the discrete Euler–Lagrange equations given by Guo and Wu [12].

Example 3.8. Let $L = \frac{1}{2} m_i \dot{q}^i(\dot{q}^i - 1) - U(q^1, \ldots, q^n)$ be a real function on $J_D^1(Z \times R^n) = (t, q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$.

Letting $\ddot{q}^\alpha = \Delta_t \dot{q}^\alpha$ and submitting L into (15), we obtain

$$m_\alpha \ddot{q}^\alpha = E_t U_{q^\alpha}. \qquad \text{no summation} \tag{16}$$

In fact

$$\begin{aligned} L_{q^\alpha} &= \Delta_s\big(\tfrac{1}{2} m_i \dot{q}^i(\dot{q}^i - 1) - U(q^1, \ldots, q^\alpha(s), \ldots, q^n)\big)\big|_{s=0} \\ &= -\Delta_s U(q^1, \ldots, q^\alpha(s), \ldots, q^n)|_{s=0} \\ &= U_{q^\alpha}, \\ L_{\dot{q}^\alpha} &= \Delta_s\big(\tfrac{1}{2} m_1 \dot{q}^1(\dot{q}^1 - 1) + \cdots + \tfrac{1}{2} m_\alpha \dot{q}^\alpha(s)(\dot{q}^\alpha(s) - 1) \\ &\quad + \cdots + \tfrac{1}{2} m_n \dot{q}^n(\dot{q}^n - 1) - U(q^1, \ldots, q^n)\big)\big|_{s=0} \\ &= \tfrac{1}{2} m_\alpha (\Delta_s \dot{q}^\alpha(s) E_s(\dot{q}^\alpha(s) - 1) + \dot{q}^\alpha(s) \Delta_s(\dot{q}^\alpha(s) - 1))|_{s=0} \\ &= m_\alpha \dot{q}^\alpha. \end{aligned}$$

If s is a continuous variable, then

$$\lim_{\varepsilon\to 0} L = \lim_{\varepsilon\to 0}\left(\tfrac{1}{2}m_i\dot{q}^i(\dot{q}^i-\varepsilon) - U(q^1,\ldots,q^n)\right) = \tfrac{1}{2}m_i\dot{q^i}^2 - U(q^1,\ldots,q^n).$$

The $\lim_{\varepsilon\to 0} L$ is nothing but the function used by Lee [3, 4] to educe the discrete Newton's equation (16). If $U_{q^\alpha} = Kq^\alpha$, where K is a constant, then

$$m_\alpha\ddot{q^\alpha} = KE_t q^\alpha. \qquad \text{no summation}$$

If s, t are the continuous variables, then it is the equation of a harmonic oscillator.

Further, the interested reader would probably benefit from a detailed description of how (15) fit in framework developed by Marsden and West *et al*.

Consider their discrete Euler–Lagrange equations [10]

$$D_2L(q^\alpha(t-1), q^\alpha(t)) + D_1L(q^\alpha(t), q^\alpha(t+1)) = 0,$$

where $D_2L(q^\alpha(t-1), q^\alpha(t)) = \frac{\partial L(q^\alpha(t-1),q^\alpha(t))}{\partial q^\alpha(t)}$, $D_1L(q^\alpha(t), q^\alpha(t+1)) = \frac{\partial D_1L(q^\alpha(t),q^\alpha(t+1))}{\partial q^\alpha(t)}$. Let

$$L_{\dot{q}^\alpha(t,\varepsilon s)}|_{s=0} = \frac{L(t, q^\alpha(t), \dot{q}^\alpha(\varepsilon,t)) - L(t, q^\alpha(t), \dot{q}^\alpha(t))}{\varepsilon},$$

$$L_{q^\alpha(t,\varepsilon s)}|_{s=0} = \frac{L(t, q^\alpha(\varepsilon,t), \dot{q}^\alpha(t)) - L(t, q^\alpha(t), \dot{q}^\alpha(t))}{\varepsilon}.$$

Since

$$\begin{aligned}\dot{q}^\alpha(t,\varepsilon s) &= \dot{q}^\alpha(t) + \varepsilon s\\ &= (q^\alpha(t+1)+\varepsilon s) - q^\alpha(t),\\ &= q^\alpha(t+1,\varepsilon s) - q^\alpha(t)\end{aligned}$$

and $\Delta_s\dot{q}^\alpha(t,\varepsilon s) = \Delta_s q^\alpha(t+1,\varepsilon s) = \varepsilon$, so

$$\begin{aligned}D_2L(q^\alpha(t), q^\alpha(t+1)) &= \lim_{\varepsilon\to 0}\left.\frac{\Delta_s L(q^\alpha(t), q^\alpha(t+1,\varepsilon s))}{\varepsilon}\right|_{s=0}\\ &= \lim_{\varepsilon\to 0}\left.\frac{\Delta_s L(q^\alpha(t), \dot{q}^\alpha(t,\varepsilon s))}{\varepsilon}\right|_{s=0}\\ &= \lim_{\varepsilon\to 0} L_{\dot{q}^\alpha(t,\varepsilon s)}\big|_{s=0}.\end{aligned}$$

Since the perturbation of $q^\alpha(t)$ is $q^\alpha(t,s) = q^\alpha(t) + s$ for all $t \in Z$,

$$\begin{aligned}\lim_{\varepsilon\to 0} L_{q^\alpha(t,\varepsilon s)}|_{s=0} &= \lim_{\varepsilon\to 0}\left.\frac{\Delta_s L(q^\alpha(t,\varepsilon s), q^\alpha(t+1,\varepsilon s))}{\varepsilon}\right|_{s=0}\\ &= \lim_{\varepsilon\to 0}\left.\frac{\Delta_s L(q^\alpha(t,\varepsilon s), q^\alpha(t+1,\varepsilon))}{\varepsilon}\right|_{s=0} + \lim_{\varepsilon\to 0}\left.\frac{\Delta_s L(q^\alpha(t), q^\alpha(t+1,\varepsilon s))}{\varepsilon}\right|_{s=0}\\ &= D_1L(q^\alpha(t), q^\alpha(t+1)) + \lim_{\varepsilon\to 0} L_{\dot{q}^\alpha(t,\varepsilon s)}\big|_{s=0}.\end{aligned}$$

Hence,

$$\begin{aligned}0 &= D_2L(q^\alpha(t-1), q^\alpha(t)) + D_1L(q^\alpha(t), q^\alpha(t+1))\\ &= \lim_{\varepsilon\to 0}(E_{-t}L_{\dot{q}^\alpha(t,\varepsilon s)} + L_{q^\alpha(t,\varepsilon s)} - L_{\dot{q}^\alpha(t,\varepsilon s)})|_{s=0}\\ &= \lim_{\varepsilon\to 0}(L_{q^\alpha(t,\varepsilon s)} - \Delta_t E_{-t}L_{\dot{q}^\alpha(t,\varepsilon s)})|_{s=0}.\end{aligned}$$

This is equivalent to the limit of (15).

4. Discrete Hamilton's mechanics

In this section, we use discrete variational equations to educe the discrete Hamilton's equations, which is equivalent to discrete Euler–Lagrange equations under the discrete Legendre transform [12]. We also obtain the discrete Noether's theorem.

4.1. Discrete Hamilton's equations

Let $Z \times R^{2n}$ be a discrete vector bundle with coordinates $\{t, p^1, \ldots, p^n, q^1, \ldots, q^n\}$. Suppose we are given a function $H = H(t, E_t p^i(t), q^i(t))$. Then, we can construct the discrete 1-form

$$\omega = d_D q^i p^i - H(t, E_t p^i, q^i) d_D t. \qquad \text{summation}$$

Setting a regular sublattice $[0, \infty)$ with a coordinate s, a discrete variation of $\sum_{t\in Z}\langle \omega, \Delta_t\rangle_D$ is given by

$$\sum_{t\in Z}\langle \widetilde{\omega}, \Delta_t\rangle_D, \tag{17}$$

where

$$\widetilde{\omega} = d_D q^i(s,t) p^i(s,t) - H(t, E_t p^i(s,t), q^i(s,t)) d_D t, \qquad \widetilde{\omega}|_{t=\{\infty,-\infty\}} = 0.$$

Now we find conditions to make the value of (17) independent of s at $s = 0$. In other words, we want to find conditions for vanishing the following expression:

$$\begin{aligned}\Delta_s \sum_{t\in Z}\langle \widetilde{\omega}, \Delta_t\rangle_D\bigg|_{s=0} &= \sum_{t\in Z}\langle i_{\Delta_s} d_D \widetilde{\omega} + d_D i_{\Delta_s}\widetilde{\omega}, \Delta_t\rangle_D\bigg|_{s=0}\\ &= \sum_{t\in Z}\langle i_{\Delta_s} d_D \widetilde{\omega}, \Delta_t\rangle_D|_{s=0} + i_{\Delta_s}\widetilde{\omega}|_{s=0,t=\infty} - i_{\Delta_s}\widetilde{\omega}|_{s=0,t=-\infty}\\ &= \sum_{t\in Z}\langle i_{\Delta_s} d_D \widetilde{\omega}, \Delta_t\rangle_D|_{s=0}.\end{aligned} \tag{18}$$

Considering the perturbation $p^i(t,s) = p^i(t) + s$ or $q^i(t,s) = q^i(t) + s$, respectively, we obtain

$$\begin{aligned} i_{\Delta_s} d_D \widetilde{\omega} &= (-\Delta_t q^i + H_{E_t p^i}(t, E_t p^i, q^i)) d_D t,\\ i_{\Delta_s} d_D \widetilde{\omega} &= (\Delta_t p^i + H_{q^i}(t, E_t p^i, q^i)) d_D t,\end{aligned} \tag{19}$$

where

$$\begin{aligned} H_{E_t p^i}(t, E_t p^i, q^i) &:= \Delta_s H(t, E_t p^i(t,s), q^i(t))|_{s=0}\\ &= H(t, E_t p^i(t,1), q^i(t)) - H(t, E_t p^i(t), q^i(t)),\\ H_{q^i}(t, E_t p^i, q^i) &:= \Delta_s H(t, E_t p^i(t), q^i(t,s))|_{s=0}\\ &= H(t, E_t p^i(t), q^i(t,1)) - H(t, E_t p^i(t), q^i(t)).\end{aligned}$$

The vanishing of (19) is equivalent to *the discrete Hamilton equations*

$$\Delta_t p^i = -H_{q^i}(t, E_t p^i, q^i), \qquad \Delta_t q^i = H_{E_t p^i}(t, E_t p^i, q^i). \tag{20}$$

If p^i and q^i satisfy equations (20), then the value of (18) vanishes.

If s is a continuous variable, then (20) is equivalent to the discrete Hamilton equations given by Guo *et al* [12].

4.2. Discrete Legendre transformation

Now we show that the discrete Hamilton's equations can also be derived from discrete Euler–Lagrange equations and vice versa, using the discrete Legendre transformation, which is given by Guo *et al* if s is continuous [12]:

$$H(t, E_t p^i, q^i) := \dot{q}^i E_t p^i - L(t, \dot{q}^i, q^i), \qquad \text{summation}$$

If only q^i or p^i or $\dot{q}^i$ depends on s respectively, then we obtain

$$H_{q^i} = -L_{q^i}, \qquad \dot{q}^i = H_{E_t p^i}, \qquad E_t p^i = L_{\dot{q}^i}, \tag{21}$$

by equaling the coefficients of $d_D s$ in equation

$$d_D H(t, E_t p^i, q^i) = d_D(\dot{q}^i E_t p^i - L(t, \dot{q}^i, q^i)).$$

From (15) and (21), we get

$$\Delta_t p^i = -H_{q^i}.$$

Combining with the third equation of (21), we induce the discrete Hamilton's equation from discrete Euler–lagrange equations.

From (20) and (21), we get

$$\Delta_t E_{-t} L_{\dot{q}^i} = L_{q^i}.$$

So we can also derive the discrete Euler–lagrange equations from discrete Hamilton's equations.

By computation, we find that the limit of the discrete Legendre transformation here is not equivalent to the transformations given by Lall *et al* [11]. Since their equations are educed from their transformations, the discrete Hamilton's equations here do not fit in their framework.

4.3. Discrete Noether's theorem

It is well known that Noether's theorem is one of the fundamental theorems in the differential case. Now we consider this theorem and show that the result here also fits in the framework developed by Marsden *et al* [10].

Now we consider the discrete 1-jet bundle of $Z \times R^n = \{t, q^1, \ldots, q^n\}$ and a function $L = L(t, q^i, \dot{q}^i)$. Consider $L(t, q^i(t,s), \dot{q}^i(t))$ a discrete variation of L. The L is said to admit the discrete variation, if

$$L(t, q^i(t,s), \dot{q}^i(t)) = L(t, q^i(t), \dot{q}^i(t)).$$

From this condition, we have

$$\begin{aligned} 0 &= \Delta_s L(t, q^i(t,s), \dot{q}^i(t))|_{s=0} \\ &= L_{q^i} \end{aligned}$$

From [15], it is equivalent to

$$\Delta_t E_{-t} L_{\dot{q}^\alpha} = 0.$$

More precisely, we have

Theorem 4.1. *Noether's theorem in the discrete case. If the function L on $J_D^1(Z \times R^n)$ admits the discrete variation*

$$q^i(s,t) : J_D^1(Z \times R^n) \to J_D^1(Z \times R^n), s \in R,$$

then the discrete Lagrangian equations corresponding to L have a first integral

$$I(q^i, \dot{q}^i) = E_{-t} L_{\dot{q}^i}.$$

Now we show how this theorem fits in the framework developed by Marsden and West *et al*.

If s is continuous, then

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0} \Delta_t E_{-t} L_{\dot{q}^i(t,\varepsilon s)|_{s=0}} \\ &= \lim_{\varepsilon\to 0} L_{\dot{q}^i(t,\varepsilon s)|_{s=0}} - E_{-t} \lim_{\varepsilon\to 0} L_{\dot{q}^i(t,\varepsilon s)|_{s=0}} \\ &= D_2 L(t, q^i(t), q^i(t+1)) - D_2 L(t, q^i(t-1), q^i(t)). \end{aligned}$$

It is nothing but the special case of their discrete Noether's theorem [10].

Future work. An important and uneasy problem is that how do the discrete solutions of equations (15) and (20) behave with respect to the analytical solutions? In order to give a satisfactory answer, we need to do much effort.

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